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Robust Model Predictive Control of Shimmy Vibration in Aircraft Landing Gears

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Shimmy vibration is one of the major concerns in the aircraft landing gear design. In this paper, the influence of structural parameters on the shimmy dynamics is analyzed based on a nonlinear dynamical model. A computationally efficient robust model predictive control law is formulated for a linear parameter varying system with guaranteed closed-loop stability. Moreover, an attempt is made to apply the proposed robust model predictive control strategy to suppress the shimmy during the taxiing and landing of an aircraft. Compared with two current robust model predictive controls, the proposed shimmy controller can effectively suppress the shimmy with more efficient computation. To verify the efficiency of the proposed algorithm, the simulation results are presented and discussed.

Nomenclature

A, B, C	=	discrete state-space matrices
A_c, B_c, C_c	=	continuous state-space matrices
A_j, B_j	=	discrete state-space matrices of jth vertex
a	=	half contact length
$C_{F\alpha}$	=	tire side force derivative
$C_{M\alpha}$	=	tire aligning moment derivative
c	=	torsional damping coefficient
e	=	wheel caster length
F	=	state feedback matrix
F_y	=	cornering force
F_z	=	vertical force
I	=	identity matrix
I_z	=	moment of inertia about z axis
J	=	total number of decision variables
J_∞	=	performance index
K	=	torsional spring rate
k	=	current time step
k_e	=	moment constant
M_z	=	tire aligning moment
M_1	=	spring moment
M_2	=	damping moment
M_3	=	total tire moment about z axis
M_4	=	tire damping moment
N	=	total row size
P	=	positive definite matrix
Q	=	weighting symmetric matrix
Q_w	=	weighting output matrix
R_w	=	weighting input matrix
T_s	=	sampling interval
$u(k)$	=	control signal
$u_{\max}$	=	control bound
V	=	taxiing velocity
$V(x)$	=	Lyapunov function
V_t	=	tire sideslip velocity
V_r	=	lateral velocity due to yaw motion of tire
$x(k)$	=	state variable
y	=	lateral deflection
z	=	system output
$z_{\max}$	=	output bound
α	=	slip angle
α_g	=	limiting slip angle for aligning moment
α_j	=	nonnegative coefficients
β	=	upper bound index
γ	=	upper bound index
δ	=	limiting slip angle
θ	=	time-varying variable
κ	=	tread width moment constant
σ	=	relaxation length
τ	=	time constant
Ψ	=	yaw angle
$\dot{\psi}$	=	yaw rate
Ω	=	polytope of vertices

I. Introduction

SHIMMY vibration is one of the major concerns in the aircraft landing gear design. Energy for this type of vibration is derived from the kinetic energy of the forward motion of the aircraft [1]. As a result, this energy initiates self-excited torsional oscillation of tires about a vertical axis that may lead to instabilities. This oscillatory motion may also be due to both the forces produced by runway surface irregularities and nonuniformities of the tires. The shimmy vibration typically has a frequency range between 10 and 30 Hz [2]. The analysis of shimmy formation can be found in [1,3,4].

Although shimmy is usually not catastrophic, it can lead to serious problems, such as excessive wear, shortened life cycle of gear parts, discomfort, safety concerns, and inconvenience for pilots and passengers. To suppress shimmy motion, shimmy damper is used in the Boeing 737 and the Airbus A-320 as a conventional preventive measure. However, as mentioned in [4], shimmy damping requirements are often conflicting with good high-speed directional control; furthermore, once the landing gear design is completed, the structural parameters for shimmy suppression cannot be changed. Hence, when external disturbances or uncertain parameters arise in the landing gear system, no further action can be taken. In some operation situations, such as worn parts, severe climate, and rough runway, active control strategy can be effective for shimmy vibration control. With the advent of high-speed and highly reliable microprocessors used in controller implementation, the idea of actively controlled landing gear has gained new momentum. Even

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though the concept of active landing gear is not new (it started in the 1970s), no production aircraft is as yet equipped with such a system, as reported in [5].

Airframe flexibility and the elastic properties of tire are two main issues considered in the landing gear modeling to the precise description of the shimmy phenomena [6]. In other words, only a fairly completed model of the structure including the tire properties could properly evaluate the stability of the landing gear system.

The tire theories are categorized into two basic groups: point contact theory and string theory. The major difference between them is the number of coordinates used to describe the tire deformation. In the first approach, the tire is considered from the kinematical behavior in the overall system. This includes Moreland's point contact theory, which assumes that the interaction between the ground and the tire takes place through a single point. The second approach, string theory, uses a physical model of the tire that is estimated by an elastic string stretched around the outer edge of the wheel [4].

In [6], modeling and simulation of landing gear systems have been presented. The input data have been carefully scrutinized, and experiments are conducted to validate the models. Studies on modeling nonlinearities, such as damping and friction characteristics, to investigate the shimmy phenomenon are becoming more prevalent recently.

There are many studies focusing on both shimmy modeling and dynamic analysis [1–3,5], whereas only a few shimmy control analyses are available in literature. In [3], active control is compared with semi-active control in aircraft suspension system, and it is pointed out that semi-active landing gear does not need large external power supply, and its implementation is simpler and more practical. Nevertheless, NASA [6] introduced a simplified model of the main landing gear of the Navy A6-Intruder equipped with an external servohydraulic system for active control on vertical damping. It was successfully demonstrated by experiments that fuselage vertical vibration level, whereas landing was reduced by a factor of 4. However, shimmy control (suppressing) is still an open problem.

In the present study, an in-depth analysis is carried out on aircraft landing gear shimmy dynamics and shimmy suppression in which the elastic string theory is used to model the tires. Based on a nonlinear aircraft landing gear model, a parametric study of its shimmy limit cycle oscillation (LCO) is made by numerical integration. The shimmy stability variation with varying caster length and taxiing velocity is also analyzed after linearizing the system. It is concluded that the linearized landing gear system is a typical linear parameter varying (LPV) system whose state-space matrices are functions of varying taxiing velocity.

The systematic LPV controller design needs some form of prediction on variation of future system dynamics [7]. Although linear quadratic regulator (LQR) can deal with LPV system within a small operating range, it cannot guarantee robust stability globally due to deficiency of any future prediction in dynamics, nor can it deal with input and output constraints systematically.

Model predictive control (MPC), or receding horizon control (RHC), is an advanced control methodology that has a significant impact on industrial control engineering [8]. It becomes increasingly popular in other control engineering problems due to the advent of powerful microprocessors. Robust control is a synthesis that optimizes worst-case performance specification and identifies worst-case parameters so long as the plant remains varying in some specified set. The uncertain model appears when system parameters are not precisely known, or may vary over a given range. In [9], a robust MPC (RMPC) synthesis is proposed that allows explicit incorporation of the description of plant uncertainties. In addition, the problem of minimizing an upper bound on the worst case is reduced to a convex optimization involving linear matrix inequalities (LMIs). It has been proved that the feasible receding horizon state feedback control robustly stabilizes the set of uncertain plants. Motivated by [9], Cuzzola et al. [10], presented a new approach based on the use of several Lyapunov functions, each of which corresponds to a different vertex of the uncertainty's polytope N. Wada et al. [11] proposed a method for synthesizing the MPC law for linear parameter varying (LPV) systems by using the parameter dependent Lyapunov function (PDLF) and claimed less conservative control performance.

The computational load involved in RMPC is a challenging problem in an online RMPC application. To cope with this drawback, a new strategy called PRMPC is proposed in the present paper to stabilize the LPV system with reduced computational load. The control objective of PRMPC is to steer the yaw angle to zero to suppress the shimmy when the landing gear system is subjected to uncertainties, which are varying taxiing velocity and disturbances. To the best of our knowledge, this is the first attempt to actively suppress shimmy vibration by using robust model predictive control with LMI approach. The proposed RMPC is compared with two present RMPCs used in landing gear shimmy control. It is verified by simulation results that the proposed RMPC stabilizes the unstable parameter varying landing gear system with guaranteed closed-loop stability, high computational efficiency, and strong disturbance rejection ability.

The paper is arranged as followed: Sec. II presents shimmy modeling and analysis for a typical landing gear. Some simulation results show the causes of shimmy oscillation and how the stability varies with taxiing velocity. Section III is devoted to propose a new robust MPC algorithm for LPV systems. The simulation results of the proposed control strategies in shimmy suppression are presented to demonstrate the control performance and the computational efficiency in Sec. IV. Comparisons with the other two MPCs are carried out to demonstrate the effectiveness of the proposed strategies. Section V presents the conclusions from the study.

II. Shimmy Modeling and Analysis

A. Shimmy Modeling

Landing gear is a multibody structure with elastic tires; hence, shimmy modeling is mostly related to the tire modeling, which involves interaction between tires and the ground. A nose landing gear shimmy model is described in [5] and is also shown in Fig. 1. This model represents a basic description of a nose gear, with a torsional degree of freedom and the tire dynamics according to the stretched string theory.

The nonlinear shimmy dynamics equations have been derived in [5,12] as given as

$$I_z\ddot{\psi} = M_1 + M_2 + M_3 + M_4 + M_5 \tag{1}$$

$$\dot{y} + \frac{V}{\sigma}y = V\psi + (e-a)\dot{\psi} \tag{2}$$

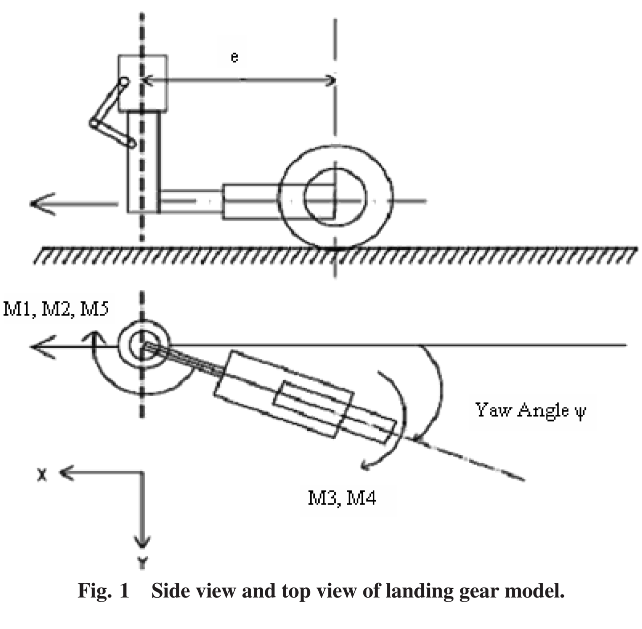


Fig. 1 Side view and top view of landing gear model.

where

$$M_1 = -c\dot{\psi}, \qquad M_2 = -K\psi, \qquad M_3 = -\frac{\kappa}{V}\dot{\psi}$$
$$M_4 = (M_z - eF_y), \qquad M_5 = k_e u \tag{1a}$$

Equation (1) represents the yaw rotation of landing gear about the vertical strut axis, whereas Eq. (2) describes the lateral deflection of tires, modeled by the elastic string theory. M_1 and M_2 account for torsional dynamics of lower parts of landing gear, whereas M_3 describes tire damping moment. M_4 is composed of lateral force F_y acting on tires with caster e as lever arm and also self-moment torques M_z about tire center. To implement the active control in the landing gear system, M_5 is introduced as the external control torque for shimmy elimination. It is proportional to the control signal u with the moment constant k_e. It is assumed that when applying such torque to steer the landing gear, the landing gear will respond with change in the yaw angle.

A rolling tire is able to transfer wheel cornering force F_y only if it rolls at an angle to the direction of travel, named slip angle α. Only when the angle increases, tires can transfer the higher wheel cornering force. The force and slip angle are, therefore, dependent on each other [13]. There are many descriptions for wheel cornering force and self-aligning moments, but the most common formulation used in the tire analysis is shown as

$$\begin{cases} F_y = C_{F\alpha}\alpha F_z & \text{for } |\alpha| \le \delta \\ F_y = C_{F\alpha}\delta F_z \sin(\alpha) & \text{for } |\alpha| > \delta \end{cases} \tag{1b}$$

where δ is limiting slip angle for tire side force.

$$\begin{cases} \frac{M_z}{F_z} = C_{M\alpha}\frac{\alpha_g}{180}\sin\left(\frac{180}{\alpha_g}\alpha\right) & \text{for } |\alpha| \le \alpha_g \\ M_z = 0 & \text{for } |\alpha| > \alpha_g \end{cases} \tag{1c}$$

with

$$\arctan(\alpha) = \frac{y}{\sigma} \tag{1d}$$

where σ is relaxation length of tire deflection mentioned in the string theory, and α_g is limiting slip angle for the aligning moment.

Elastic string theory is used in the modeling of tire motion Eq. (2). The formation of a slip angle may result from either of two fundamental motions, that is, pure yaw or pure sideslip. A tire rolls in pure yaw only when the yaw angle ψ is allowed to vary, and the lateral deflection y is held at zero. On the contrary, in pure sideslip, the lateral deflection varies as the yaw angle is held at zero. The deflection of tire is due to ground forces acting on the tire footprint, and these ground forces (or moments) are transmitted through the tire to the wheel. According to the elastic string theory, the lateral deflection y of the leading contact point of tire with respect to tire plane can be described as a first order differential equation with time constant τ ($\tau = \sigma/V$) [12]. The parameter σ is relaxation length, which is defined as the ratio of slip stiffness to longitudinal force stiffness. Tire sideslip velocity V_t can be expressed as

$$V_t = \dot{y} + \frac{y}{\tau} \tag{2a}$$

The yaw motion of the tire leads to a lateral yaw velocity V_r, which is similar to V_t and can be approximated as in [14]:

$$V_r = V\psi + (e - a)\dot{\psi} \tag{2b}$$

When the tire rolls on the ground, the following equation is satisfied:

$$V_t = V_r \tag{2c}$$

Substituting Eqs. (2a) and (2b) into Eq. (2c), one can obtain Eq. (2).

Table 1 Parameter values in model

Symbol	Value	Unit
A	0.1	m
K	10,000	$\mathrm{N \cdot m/rad}$
$C_{F\alpha}$	20	1/rad
$C_{M\alpha}$	−2	m/rad
F_z	9000	N
Δ	5	deg
E	0.1	m
I_z	1.0	$\mathrm{Kg \cdot m^2}$
C	0…50	$\mathrm{N \cdot m \cdot s/rad}$
V	0…80	m/s
α_g	10	deg
K	270	$\mathrm{N \cdot m^2/rad}$
Σ	0.3	m
k_e	10,000	$\mathrm{N \cdot m/Volt}$

B. Shimmy Existence

The values of all parameters in Eqs. (1) and (2) are given in Table 1. The external moment constant k_e depends on output and input relationship of external actuator. Referring to torsional spring rate K of 1000 $\mathrm{N \cdot m/rad}$, without loss of generality, the moment constant k_e is chosen as 10,000 $\mathrm{N \cdot m/volt}$. The nonlinear differential Eqs. (1) and (2) can be solved by numerical integration. Here, the initial value of disturbed yaw angle is set to 0.1 rad, torsional damping constant c is 10 $\mathrm{N \cdot s/rad}$, and taxiing velocity is 60 m/s. Shimmy vibration in lateral direction in Fig. 2 and yaw limit cycle in Fig. 3 are clearly observed.

C. Stability Variation Analysis

F_y and M_z/F_z are linearly proportional to sideslip angle α within a small range. Based on this assumption, the nonlinear dynamic system is linearized using Taylor series expansion and rearranged as state-space equations as in Eq. (3).

$$\dot{x} = A_c x + B_c u \qquad z = C_c x \tag{3}$$

where

$$x = \begin{pmatrix} \psi \\ \dot{\psi} \\ y \end{pmatrix}, \qquad A_c = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-K}{I_z} & \frac{-c}{I_z} + \frac{\kappa}{VI_z} & \frac{F_z}{I_z\sigma}(C_{M\alpha} - eC_{F\alpha}) \\ V & e - a & -\frac{V}{\sigma} \end{pmatrix},$$

$$B_c = \begin{Bmatrix} 0 \\ k_e \\ 0 \end{Bmatrix}, \qquad C_c = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T$$

Hence, the output of the system is the yaw angle ψ.

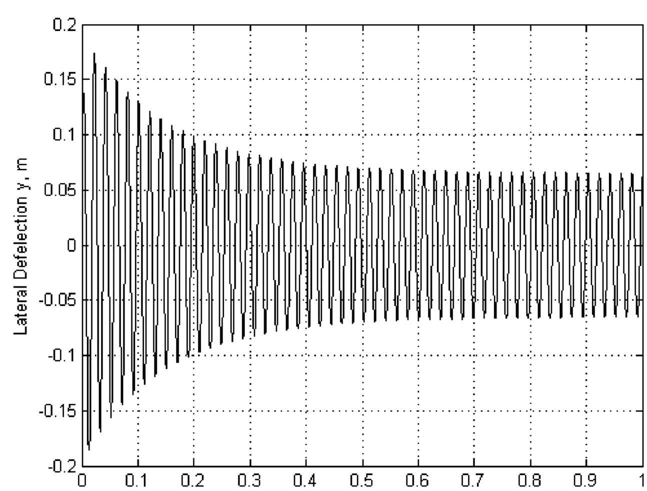


Fig. 2 Time history of lateral deflection shimmy dynamics $\Psi(0) = 1$ (rad), $K = 10$ (N/m), $V = 60$ (m/s).

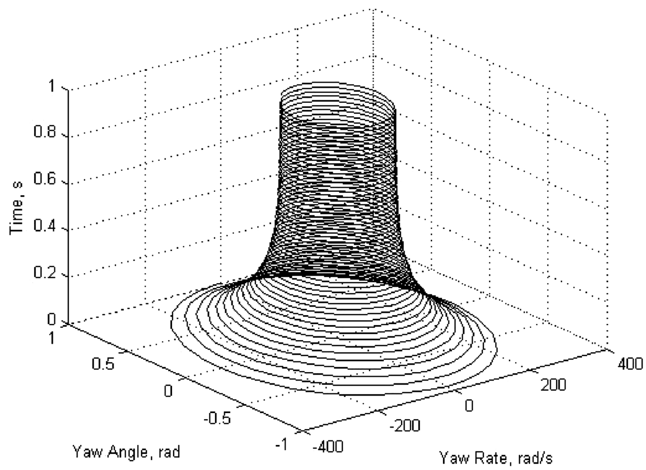


Fig. 3 Phase plot history of yaw angle. Phase plot of shimmy $\psi(0) = 1$ (rad), $k = 10$ (N/m), $V = 60$ (m/s).

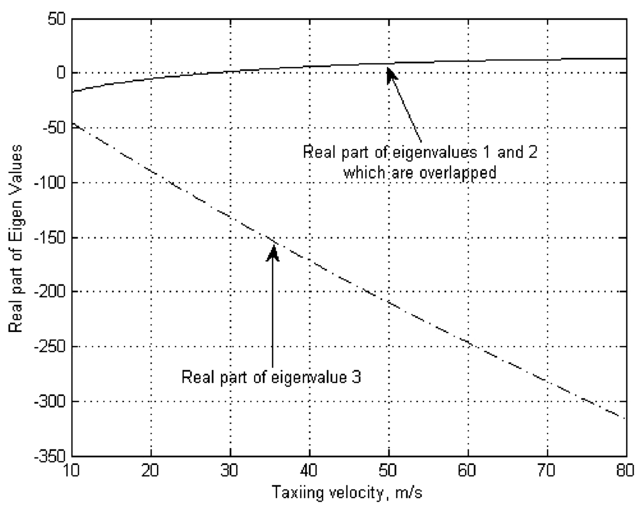


Fig. 4 Real part of eigenvalues, variation with taxiing velocity V.

The inherent varying parameter in the landing gear system is taxiing velocity V during landing and takeoff. Taxiing velocity is critical in the shimmy analysis [1,15]. As shown in dynamics simulation, lower taxiing speed leads to higher stability and the landing gear becomes stable at the lower speed of 10 knots (5.144 m/s). It is also reported in [4] that the shimmy vibrations increase with increased velocity. Before the aircraft touches down, forward velocity is supposed to be lower than 150 knots (77.2 m/s).

Varying parameters such as the caster length e are often considered by landing gear designers and now is extended to stability analysis. By varying parameters (V) and (e), system eigenvalues vary and consequently affect system stability. The linear model (3) has a pair of complex roots λ_1, λ_2 and a stable real root λ_3. According to Fig. 4, landing gear is an unstable system for velocities above 25 m/s, and, based on Fig. 5, for caster lengths between 0.05 and 0.15 m, it is marginally stable due to positive real parts of the complex roots. However, when the velocity goes below 25 m/s, the system becomes stable. For controller design, the taxiing velocity range is set between 80 and 20 m/s. The objective of robust control design is then to introduce a state feedback controller to stabilize this unstable landing gear system when it is subjected to vary taxiing velocity and external disturbances.

III. Robust Model Predictive Controller Design

In this section, starting from concept of linear parameter varying system, a new robust MPC that asymptotically stabilizes the system will be proposed.

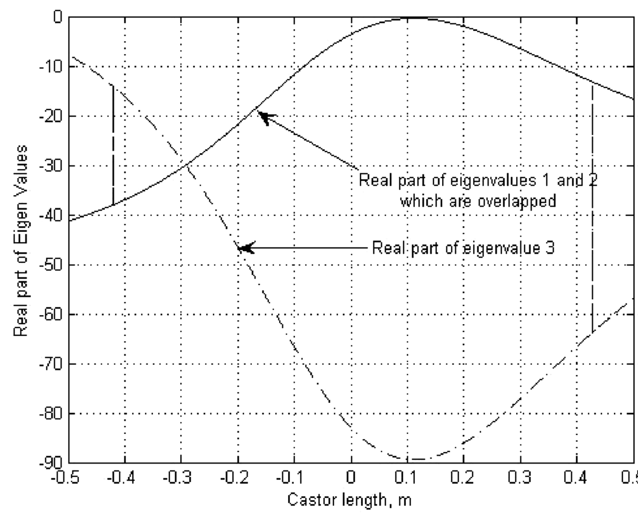


Fig. 5 Real part of eigenvalues, variation with caster length e.

A. Linear Parameter Varying System

The LPV system is a linear system with time varying parameters. Robust model predictive control is a powerful design tool based on horizontal receding control and robust control theories that deal with such system. As presented in Sec. II, the linearized landing gear model (3) is a LPV system whose dynamics inherently depends on taxiing velocity V. To introduce robust state feedback control, a general LPV discrete-time dynamic model is considered as following:

$$x(k+1) = A(\theta)x(k) + Bu(k) \qquad z(k) = Cx(k) \tag{4}$$

where A is a matrix function of parameter (θ), which stands for any varying parameter in $A(\theta)$ (e.g., taxiing velocity V in landing gear system, but not limited to single varying parameter). It is assumed that the system $[A(\theta), B]$ varies in a polytope Ω (convex hull) of vertices $[A_1|B_1], [A_2|B_2], \cdots, [A_L|B_L]$, denoted as

$$[A(k)|B(k)] \in \Omega, \quad \forall\ k > 0 \tag{5}$$

where $\Omega := Co\{[A_1|B_1], [A_2|B_2], \cdots, [A_L|B_L]\}$ represents convex hull, that is, $\forall k$, there exist L nonnegative coefficients α_j ($j = 1, 2, \ldots L$) such that

$$[A(\theta), B] = \sum_{j=1}^{L} \alpha_j [A_j, B_j] \tag{6}$$

where

$$\sum_{j=1}^{L} \alpha_j = 1, \qquad \alpha_j \geq 0, \quad \forall\ j \in [1, L]$$

The present study addresses the class of LPV systems in which the state-space matrices depend on the time-varying parameter vector (θ) so that the system $[A(\theta), B]$ varies within a polytope. This type of system has been studied for many years in an LMI framework [16]. In the following, it is assumed that the state variables $x(k)$ are fully measurable, and $x(k|k) = x(k)$. Consequently, a state feedback control synthesis procedure is proposed based on LMI techniques.

B. Robust Model Predictive Control Design

Consider the LPV system (4) subjected to the input and output constraints

$$\|u(k+i|k)\|_2^2 \leq u_{\max}, \qquad \|z(k+i|k)\|_2^2 \leq z_{\max} \qquad (i \geq 0) \tag{7}$$

where $u_{\max}$ is the upper bound on input $u(k+i|k)$, and $z_{\max}$ is the upper bound on output $z(k+i|k)$. The control objective is to find a control law $u(k)$ so that the variable state $x(k)$ can be steered to zero in desirable time. The robust MPC is designed to optimize the robust objective performance [9]:

$$\min_{u(k+i|k)} \max_{[A(k+i),B(k+i)]\in\Omega} J_\infty(k) \tag{8}$$

where

$$J_\infty(k) = \sum_{k=0}^{\infty}[x(k+i|k)^T Q_w x(k+i|k)] + [u(k+i|k)^T R_w u(k+i|k)]$$

$Q_w > 0$ and $R_w > 0$ are two weighting matrices. In the RMPC, the Lyapunov function is defined as

$$V(x) = x(k|k)^T P x(k|k)$$

by $P := \gamma Q^{-1} > 0$. By satisfying the following inequality, one can achieve the optimal performance objective [eq. (8)] [9].

$$V[x(k+1|k)] - V[x(k|k)] \leq -[x^T(k|k) Q_w x(k|k) + u^T(k|k) R_w u(k|k)] \tag{9}$$

Summing up this inequality from $k = 0$ to ∞ and $x(\infty|k) = 0$ yields the upper bound of the Lyapunov function.

$$J_\infty(k) \leq V[x(k|k)] < \gamma \tag{10}$$

Supposing that the uncertainty set Ω is defined by a polytope as in Eq. (5), the state feedback matrix F in the control law is $u(k+i|k) = F(k)x(k+i|k)$, $i \geq 0$ that minimizes the upper bound on the robust performance objective function at sampling time k given by

$$F(k) = YQ^{-1} \tag{11}$$

where $Q > 0$ and Y are obtained from the solution of the following linear objective minimization problem [9].

$$\min_{\gamma,Q,Y} \gamma \tag{12}$$

subjected to

$$\begin{bmatrix} 1 & x^T(k) \\ x(k) & Q \end{bmatrix} \geq 0 \tag{13}$$

$$\begin{bmatrix} Q & (A_j Q + BY)^T & Q(Q_w^{1/2})^T & Y^T(R_w^{1/2})^T \\ A_j Q + BY & Q & 0 & 0 \\ Q_w^{1/2} Q & 0 & \gamma I & 0 \\ R_w^{1/2} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0 \tag{14}$$

$$\begin{bmatrix} u_{\max}^2 & Y \\ Y^T & Q \end{bmatrix} \geq 0 \tag{15}$$

$$\begin{bmatrix} z_{\max}^2 Q & C^T(A_j Q + BY)^T \\ (A_j Q + BY)C & z_{\max}^2 I \end{bmatrix} \geq 0 \tag{16}$$

Under this designed closed-loop feedback law, the LPV system (4) is stabilized, and the state variable $x(k)$ is steered to zero. At each sampling time, an optimal upper bound on the worst-case performance cost over the infinite horizon is obtained by forcing a quadratic function of the state to decrease by at least the amount of the worst-case performance cost at each prediction time. Such online step-by-step optimization can lead to asymptotically stable evolution. But for the real-time application, especially for the shimmy suppression in the landing gear system, the computational efficiency of LMI is very critical to guarantee the practical implementation of RMPC. To improve the computational efficiency, we need to sacrifice the optimal performance to the computational load at each step. A new robust MPC is proposed in the following section.

C. Proposed Computationally Efficient Robust Model Predictive Control

To alleviate the computational load in RMPC, an attempt is made to reduce the dimension of matrices in LMI [Eqs. (13–16)]. Inequality (14) involves the most computational load because it has to be satisfied by every vertex (A_j, B_j). It is found that the matrices in rows 3 and 4 in inequality (14) are directly related to the robust performance index, which appears at the right-hand side of inequality (9). The proposed RMPC will trade the optimal performance with the computational load. The main objective of controller design is to guarantee the asymptotic convergence of the system.

To accomplish this, a Lyapunov function $V(x) = x(k|k)^T P_k x(k|k)$ is defined, where $P_k := \beta Q_k^{-1} > 0$ is a positive definite matrix that will be obtained by solving the optimal problem at current time (k). To guarantee the asymptotic stability of the closed-loop system, we must define a Lyapunov function $V(x)$, a strictly decreasing function, for the closed-loop system.

$$V[x(k+1|k+1)] - V[x(k|k)] < 0 \tag{17}$$

It is equivalent to the following inequality:

$$x(k+1|k+1)^T P_{k+1} x(k+1|k+1) - x(k|k)^T P_k x(k|k) < 0 \tag{18}$$

where the measured state variables at time $k+1$ ($x(k+1|k+1)$) are assumed to be equal to the predicted state variables $x(k+1|k)$. To guarantee that inequality (18) holds, the following inequality must be ensured:

$$x(k+1|k+1)^T P_{k+1} x(k+1|k+1) < x(k+1|k+1)^T P_k x(k+1|k+1) \tag{19}$$

By using the definition of P_k, we can write the inequality in terms of Q_{k+1} and Q_k:

$$x(k+1|k+1)^T \beta Q_{k+1}^{-1} x(k+1|k+1) < x(k+1|k+1)^T \beta Q_k^{-1} x(k+1|k+1) \tag{20}$$

The robust MPC is designed to make the matrix Q_{k+1}^{-1} at time $k+1$ to be smaller than Q_k^{-1} at time k. This is equivalent to the case in which the matrix Q_k^{-1} is smaller than the matrix Q_{k-1}^{-1}, illustrated as follows:

$$Q_k^{-1} < Q_{k-1}^{-1} \tag{21}$$

It is rewritten in the following form by adding an upper bound βI to matrix Q_k:

$$Q_{k-1} < Q_k < \beta I \tag{22}$$

Furthermore, to guarantee that inequality (18) is satisfied, the right-hand side of inequality (20) needs to satisfy the following inequality:

$$x(k+1|k+1)^T P_k x(k+1|k+1) = x^T(k+1|k) \beta Q_k^{-1} x(k+1|k) \leq x^T(k|k) \beta Q_k^{-1} x(k|k) \tag{23}$$

Assume B_j for each vertex $[A_j, B_j]$ is fixed, and $B_j = B$ for $j = 1, 2, \ldots, L$. Substituting $x(k+1|k) = A(\theta)x(k|k) + Bu(k)$ and feedback control $u(k) = F(k)x(k|k)$ into the inequality yields

$$x^T(k|k)[A(\theta) + BF(k)]^T \beta Q_k^{-1} [A(\theta) + BF(k)] x(k|k) - x^T(k|k) \beta Q_k^{-1} x(k|k) \leq 0 \tag{24}$$

With the definition of feedback control gain in Eq. (11), we have the following inequality:

$$\left(\sum_{j=1}^{L}\alpha_j A_j + BYQ_k^{-1}\right)^T \beta Q_k^{-1}\left(\sum_{j=1}^{L}\alpha_j A_j + BYQ_k^{-1}\right) - \beta Q_k^{-1} \le 0 \quad (25)$$

Thus, this inequality can be rewritten as the following inequality by using the polytope uncertainty (6) and Schur complement [16]:

$$\sum_{j=1}^{L}\alpha_j \begin{bmatrix} Q_k & (A_jQ_k + BY)^T \\ A_jQ_k + BY & Q_k \end{bmatrix} \ge 0 \quad (26)$$

Hence, this inequality holds if the following L inequalities hold.

$$\begin{bmatrix} Q_k & (A_jQ_k + BY)^T \\ A_jQ_k + BY & Q_k \end{bmatrix} \ge 0, \qquad j = 1, 2, \cdots L \quad (27)$$

In summary, the proposed robust MPC for LPV system (4) subjected to the input and output constraints is posed as the following minimization problem:

$$\min_{\beta, Q_k, Y} \beta \quad (28)$$

subjected to

$$\begin{bmatrix} Q_k & (A_jQ_k + BY)^T \\ A_jQ_k + BY & Q_k \end{bmatrix} \ge 0, \qquad j = 1, 2, \cdots L \quad (29)$$

$$Q_{k-1} < Q_k < \beta I \quad (30)$$

$$\begin{bmatrix} u_{\max}^2 & Y \\ Y^T & Q \end{bmatrix} \ge 0 \quad (31)$$

$$\begin{bmatrix} z_{\max}^2 Q & C^T(A_jQ + BY)^T \\ (A_jQ + BY)C & z_{\max}^2 I \end{bmatrix} \ge 0 \quad (32)$$

The feedback control gain is obtained as $F(k) = YQ_k^{-1}$, and the control signal is calculated as $u(k+i|k) = F(k)x(k+i|k)$, $i \ge 0$. The detailed step-by-step control algorithm is summarized as follows.

The algorithm of the proposed MPC. Given an initial state x_0, the controller for LPV system is implemented as follows:

1) $k = 0$, compute Q_0 and Y_0 by minimizing problem solving (12). Subjected to Eqs. (13–16) with initial condition x_0, input $u_{\max}$ and output constraints $z_{\max}$, save the corresponding Q_0, Y_0.

2) $k = 1$, find Y and Q_k by solving minimization problem (28). Subjected to Eqs. (29–32), calculate the feedback gain as $F(k) = YQ_k^{-1}$ and control input as $u(k) = F(k)x(k)$.

3) Apply the control input $u(k)$ to LPV system $k = 1, \ldots n$; go to step 2.

Remark 1: The algorithm starts with solving the time-consuming minimization problem subjected to LMIs (13–16) before the iteration and then switches to the proposed RMPC. The optimization problem outside the loop is solved to obtain the initial Q_0, which will serve as the lower bound of next step Q_1 in the loop. In each step, the matrix Q_k is solved and used for feedback gain $F(k)$ computation. The step-by-step optimization of this problem can lead to asymptotically stable evolution.

Remark 2: The proposed algorithm is based on RMPC and invariant ellipsoid concept [9]. In step 1, the upper bound (γ_0) of Lyapunov function for the initial state variables x_0 is minimized, and the inequality $x(0)^T P_0 x(0) \le \gamma_0$ is held. The obtained subset of state space $x \in R^{n_x}$, $\varepsilon = \{x | x^T Q_0^{-1} x \le 1\}$ becomes invariant ellipsoidal set of state variables. And γ_0 also becomes the upper bound of Lyapunov function for system (4) by solving the proposed optimization problem in step 2.

Remark 3: According to LMI optimization theory [16], the fastest interior point algorithm's computational effort grows with (NJ^3), where N is the total row size of LMIs, and J is the total number of decision variables. Consider the minimization problem (28). The total row size of LMIs (29–32) has been reduced compared with those of LMIs (13–16) in each iteration. Thus, the computational load is significantly reduced.

In summary, consider a LPV system (4) with $(A(\theta), B)$ varying in a polytope Ω (convex hull) of vertices $[A_1|B_1], [A_2|B_2], \cdots, [A_L|B_L]$. Assume that the system is subjected to input and output constraints [Eq.(7)]. The state feedback matrix $F(k)$ in the control law $u(k) = F(k)x(k)$ is given by $F(k) = YQ_k^{-1}$, where $Q_k > 0$ and Y are obtained from the solution of the linear objective minimization problem (28), subjected to LMIs (29–32). Then the obtained control law robustly stabilizes the closed-loop system asymptotically.

IV. Robust Shimmy Control for Landing Gear System

The proposed RMPC algorithm is applied in landing gear shimmy control. Consider the linearized shimmy dynamics, Eq. (3). The objective of shimmy control is to asymptotically suppress yaw vibration with low overshoot and short settling time during landing process and to robustly stabilize the system despite the taxiing velocity varying from 80 to 20 m/s.

A. Linear Parameter Varying Landing Gear Model

To apply the proposed RMPC for shimmy suppression, the linearized landing gear model is discretized by using Bilinear (Tustin) approximation by sampling interval T_s.

$$x(k+1) = A(V)x(k) + Bu(k) \qquad z(k) = Cx(k) \quad (33)$$

Suppose that the control input constraint is $\|u(k+i|k)\|_2 \le u_{\max}$, and the output constraint is $\|z(k+i|k)\|_2 \le z_{\max}$. These constraints must always be satisfied in the process of control design. The index β is to be minimized at every sampling interval to guarantee the robust stability when the system is subjected to the varying taxiing velocity and the external disturbance.

B. Polytope Construction

Note that two parameters V and $1/V$ appear in Eq. (4). The involved LMI's computation in RMPC is identified as a convex optimization. Hence, before starting to design control law of RMPC, a convex polytope must be constructed. In this paper, the method of polytope construction is illustrated in Fig. 6: Line 1–4 is the connection between maximum and minimum of velocity; line 1–2 is tangent to hyperbola at point 1 (minimum velocity 20 m/s); line 3–4 is tangent to hyperbola at point 4 (maximum velocity 80 m/s); and line 2–3 is parallel to line 1–4.

This polytope covers the whole range of varying parameters $(V, 1/V)$, as shown in Fig. 6. According to convex set theory, any

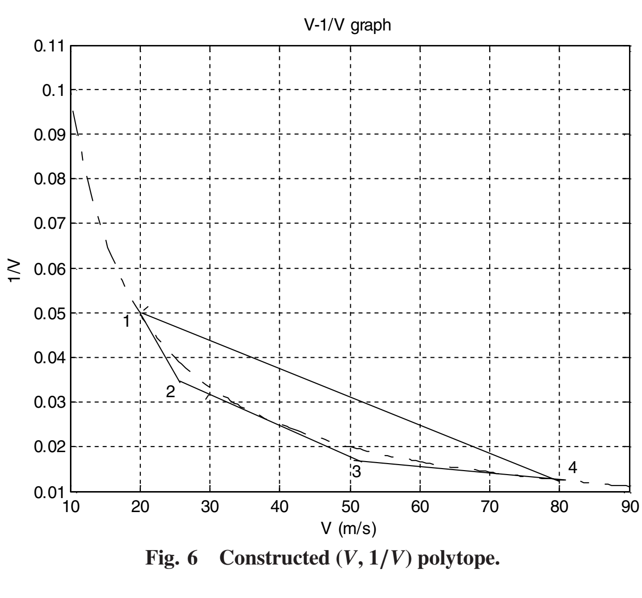


Fig. 6 Constructed $(V, 1/V)$ polytope.

Table 2 Shimmy control system design parameters

Discretization method	Bilinear (Tustin) approximation
System sampling time	0.005 s
Maximum control input	0.5 V
Maximum system output	1 rad
Initial condition: yaw angle (ψ)	0.1 rad
Initial condition: lateral deflection (y)	0.05 m
Weighting input coefficient R_w	1
Weighting output matrix Q_w	$0.01 \times I$

point on the curve $(V, 1/V)$ can be represented by linear combination of four vertices: P_1, P_2, P_3, and P_4. Therefore, four vertices formed by P_1, P_2, P_3, and P_4 are illustrated as: $P_1(20, 0.05)$, $P_2(26.48, 0.037)$, $P_3(54.07, 0.0185)$, and $P_4(80, 0.0125)$. The system (A, B) varies in a polytope Ω (convex hull) of vertices $[A_1(P_1)|B], [A_2(P_2)|B], \cdots, [A_4(P_4)|B]$, where

$$A = \sum_{j=1}^{4} A_j(p_j)\alpha_j; \qquad \sum_{j=1}^{4} \alpha_j = 1; \qquad \alpha_j \geq 0$$

$$A_1 = \begin{bmatrix} 1.00 & 0.0050 & 0 \\ -500.00 & 0.8325 & -600.00 \\ 0.10 & 0 & 0.6667 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.00 & 0.0050 & 0 \\ -500.00 & 0.8490 & -600.00 \\ 0.1324 & 0 & 0.5586 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.00 & 0.0050 & 0 \\ -500.00 & 0.8750 & -600.00 \\ 0.2704 & 0 & 0.0988 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1.0000 & 0.0050 & 0 \\ -500.00 & 0.8831 & -600.00 \\ 0.4000 & 0 & -0.3333 \end{bmatrix}$$

C. Simulation and Comparison

To implement the proposed RMPC, it is necessary to set several important design parameters such as weight matrices for the robust MPC (R_w, Q_w), sampling interval (T_s), control input, and output constraints $(u_{\max}, z_{\max})$, etc. Decreasing control maximum can lead to smoother control. However, it will lead to infeasibility of the convex optimization algorithm. In addition, the choice of sampling interval is critical to the success of optimization at each step. The sampling interval is short enough that the sample frequency is 4 times greater than the system bandwidth. However, it should be long enough to accommodate the computation involved in the robust MPC algorithm. All tuned parameters have to assure feasible LMIs solution and are listed in Table 2.

D. Simulation Environment

The hardware configuration of the computer is Dell Workstation PWS370 with Intel Pentium 4 CPU 2.80 GHz, 512 MB RAM, and hard disk 250 GB, with 36 GB free space. The operating system is Windows XP (SP2), NTFS file system. The algorithms are implemented in MATLAB® 7.0.1(R14) SP1.

E. Control Performance

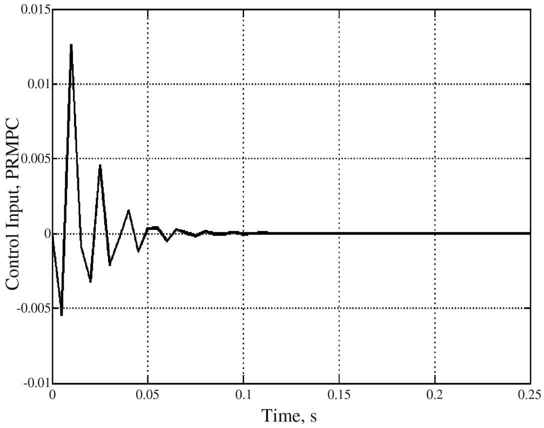
In shimmy control design, all involved LMIs are effectively solved by Matlab + YALMIP Toolbox. Two other RMPCs are designed for performance comparison. One is proposed by Kothare et al. in [9] (called KRMPC), and another is an improved method proposed by Cuzzola et al. (called CRMPC) in [10]. As it can be seen from Sec. III, the proposed RMPC (called PRMPC) is also an improved RMPC based on [9].

The taxiing velocity (V) is changed from 80 to 20 m/s within 600 steps (3 s). Three methods can suppress the shimmy within first 50 steps (0.25 s) and stabilize yaw angle at zero during the simulation process (3 s). To demonstrate the control performance clearly, we set the simulation steps as 50 (simulation time 0.25 s) and compare three methods. The simulation results are shown in Figs. 7–9, and the comparison of three RMPCs are summarized in Table 3.

Because the minimization problem in PRMPC is solved in a simpler and smaller-scale convex optimization, much less computational effort is needed. Hence, the comparison results show that the proposed RMPC is the fastest among those three RMPCs. The computational time is reduced by 2.5 times. Even though the computational efficiency of RMPC is improved significantly compared with the other two methods, the average computational time per loop of RMPC is larger than the sampling interval (0.005 s). In the real-time implementation, a more powerful microprocessor is needed to further reduce the computational time.

The starting value of cost function γ depends on the system dynamics, initial condition, and algorithm. According to [8], the problem formulation of RMPC tends to be somewhat conservative. From Table 3, CRMPC starts from a value less than KRMPC and PRMPC at the cost of heavier computation burden, which leads to longer total computational time and average computational time per loop.

From Figs. 7–9, one may observe that the state convergence in PRMPC is almost as good as in CRMPC, although KRMPC introduces another matrix variable, referring to [9,10]. KRMPC exhibits no overshoot and a good convergence with 0.1 s of settling time. However, the convergence in PRMPC with less than 0.1 s settling time is similar to that in KRMPC, but with less computational time as shown in Table 3.



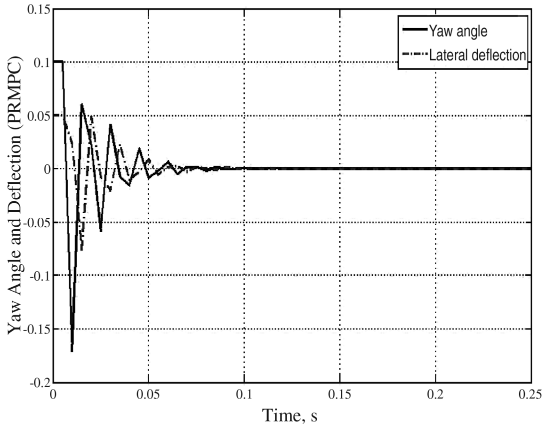


Fig. 7 Control input and state history of PRMPC.

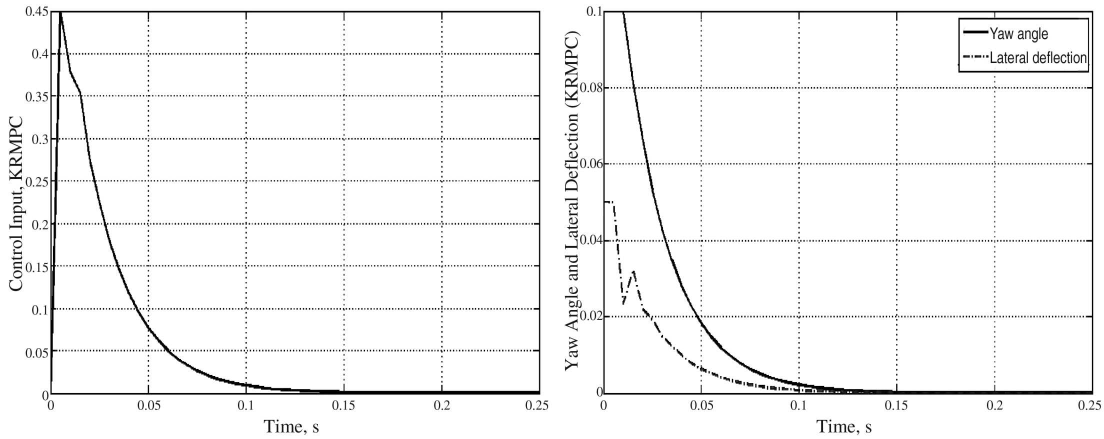


Fig. 8 Control input and state history of KRMPC.

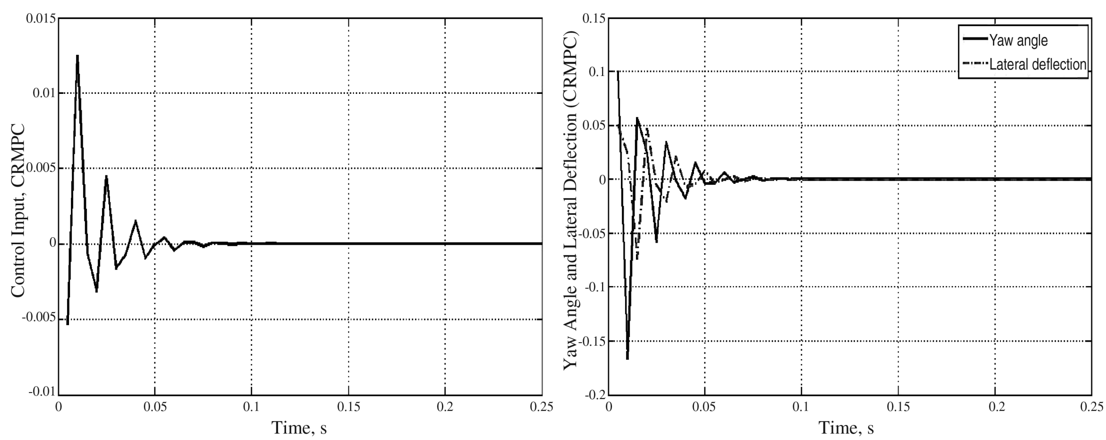


Fig. 9 Control input and state history of CRMPC.

F. Disturbance Rejection

Many control systems in practice are subjected to disturbances. The common external disturbances for landing gear arise from crosswind and rough runway. When the aircraft is experiencing any external disturbance (i.e., potholes, cracks, unevenness on runway, etc.), the landing gear body should not have large shimmy, and the shimmy oscillations should dissipate as quickly as possible.

In the following simulations, the landing gear is assumed to taxi along the runway with varying taxiing velocity from 80 to 20 m/s within 3 s. In this simulation, the runway disturbance is approximated by a step input which lasts for 10 steps (0.05 s). The simulation results are shown in Figs. 10–12. Three controllers can deal with disturbance and stabilize the systems. The proposed controller (PRMPC) and CRMPC have much less overshoot and shorter settling time than KRMPC controller.

Remark 4: To the best of our knowledge, this is the first attempt to conceptually apply RMPC for shimmy suppression. There are certainly several implementation issues involved in active shimmy control system. The speed of response of the external actuator may not be high enough to deliver the control action. The introduction of the external actuators may affect the reliability of landing gear system. Incorporating shimmy control system with advanced brake control system (ABCS) may be a possible approach. The measurement noise in the state variables has not been considered in the study.

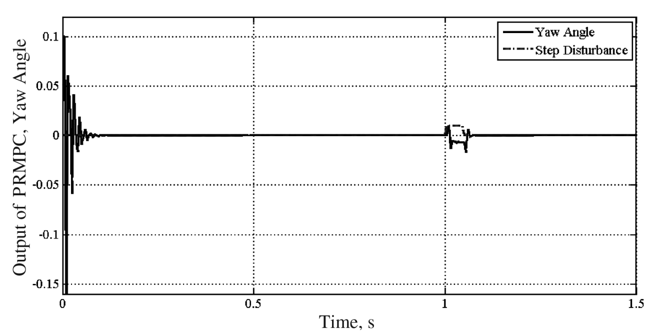


Fig. 10 PRMPC performances with step disturbance.

V. Conclusions

In this paper, an active control strategy is proposed for landing gear shimmy control using RMPC. Based on the landing gear dynamic model, shimmy limit cycle oscillations are observed, and

Table 3 Comparison of three RMPCs

Control method	KRMPC	CRMPC	PRMPC
Starting value of upper bound index	5537.6(γ)	1146.6(γ)	5537.6(β)
Total computational time, s	63.5	465.3	25.8
Average computational time per loop, s	1.27	9.04	0.52

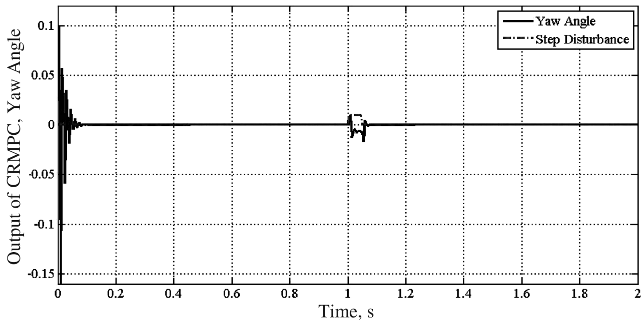


Fig. 11 CRMPC performances with step disturbance.

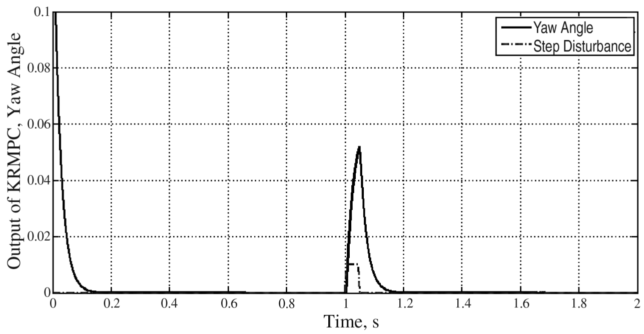


Fig. 12 KRPMC performances with step disturbance.

stability variations are simulated. A new RMPC algorithm is proposed to improve the computational efficiency in the optimization problem. Introducing RMPC state feedback, the control law calculated by step-by-step optimization can asymptotically stabilize the LPV system with disturbance rejection ability. The simulation results demonstrate that the proposed RMPC can effectively suppress shimmy for the nominal operation range of aircraft during landing when the taxiing velocity changes from 80 to 20 m/s. The proposed PRMPC improves the computational efficiency while achieving the control performance similar to that of two current RMPC algorithms.

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